

Central limit theorems for order parameters of the Gardner problem

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Abstract

Fluctuations of the order parameters of the Gardner model for any $\alpha < \alpha_c$ are studied. It is proved that they converge in distribution to a family of jointly Gaussian random variables.

1 Introduction and Main Results

The Gardner model was introduced in [G] to study the typical volume of interactions between each pair of N Ising spins which solve the problem of storing a given set of p random patterns $\{\xi^{(\mu)}\}_{\mu=1}^p$. The components $\xi_i^{(\mu)}$ of the patterns are taken usually to be independent random variables with zero mean and variance 1. After a simple transformation this problem is reduced to the analysis of the asymptotic behaviour of the random variable

$$\Theta_{N,p}(k) = \sigma_N^{-1} \int_{(\mathbf{J}, \mathbf{J})=N} d\mathbf{J} \prod_{\mu=1}^p \theta(N^{-1/2}(\xi^{(\mu)}, \mathbf{J}) - k), \quad (1.1)$$

where the function $\theta(x)$, as usually, is zero in the negative semi-axis and 1 in the positive and σ_N is the Lebesgue measure of N -dimensional sphere of radius $N^{1/2}$. Then, the question of interest is the behaviour of $\frac{1}{N} \log \Theta_{N,p}(k)$ in the limit $N, p \rightarrow \infty$, $\frac{p}{N} \rightarrow \alpha$. Gardner [G] had solved this problem by using the so-called replica trick, which is completely non-rigorous from the mathematical point of view but sometimes very useful in the physics of spin glasses (see [M-P-V] and references therein).

She obtained that for any $\alpha < \alpha_c(k)$, where

$$\alpha_c(k) \equiv \left(\frac{1}{\sqrt{2\pi}} \int_{-k}^{\infty} (u+k)^2 e^{-u^2/2} du \right)^{-1}, \quad (1.2)$$

the following limit exists

$$\lim_{N,p \rightarrow \infty, p/N \rightarrow \alpha} \frac{1}{N} E\{\log \Theta_{N,p}(k)\} = \min_{0 \leq q \leq 1} \left[\alpha E \left\{ \log H \left(\frac{u\sqrt{q} + k}{\sqrt{1-q}} \right) \right\} + \frac{1}{2} \frac{q}{1-q} + \frac{1}{2} \log(1-q) \right], \quad (1.3)$$

where u is a Gaussian random variable with zero mean and variance 1, $H(x)$ is defined as

$$H(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt \quad (1.4)$$

and here and below we denote by the symbol $E\{\dots\}$ the averaging with respect to all random parameters of the problem and also with respect to u . And $\frac{1}{N} \log \Theta_{N,p}(k)$ tends to minus infinity for $\alpha \geq \alpha_c(k)$.

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In the paper [S-T2] (see also [S-T3]) we have studied the Gardner problem in a regular mathematical way and proved that for any $\alpha < \alpha_c$ formula (1.3) is valid while for $\alpha > \alpha_c$ any $\frac{1}{N}E\{\log \Theta_{N,p}(k)\} \rightarrow -\infty$, as $N, p \rightarrow \infty, p/N \rightarrow \alpha$. We studied the case $\xi_i^{(\mu)} = \pm 1$, but of course the same results are valid for any distribution $\xi_i^{(\mu)}$, if $E|\xi_i^{(\mu)}|^4 < \infty$.

To obtain this results we introduced an intermediate "modified" Hamiltonian depending on the parameters $\varepsilon > 0, z > 0$.

$$\mathcal{H}(\mathbf{J}, k, h, z, \varepsilon) \equiv - \sum_{\mu=1}^p \log H \left(\frac{k - (\boldsymbol{\xi}^{(\mu)}, \mathbf{J}) N^{-1/2}}{\sqrt{\varepsilon}} \right) + (\mathbf{h}, \mathbf{J}) + \frac{z}{2}(\mathbf{J}, \mathbf{J}), \quad (1.5)$$

where the function $H(x)$ is defined in (1.4) and $\mathbf{h} = (h_1, \dots, h_N)$ is an external random field with independent Gaussian h_i with zero mean and variance 1.

The partition function for this Hamiltonian is

$$Z_{N,p}(k, h, z, \varepsilon) = \sigma_N^{-1} \int d\mathbf{J} \exp\{-\mathcal{H}_\varepsilon(\mathbf{J}, k, h, z, \varepsilon)\}. \quad (1.6)$$

We denote also by $\langle \dots \rangle$ the corresponding Gibbs averaging and

$$f_{N,p}(k, h, z, \varepsilon) \equiv \frac{1}{N} \log Z_{N,p}(k, h, z, \varepsilon). \quad (1.7)$$

In [S-T2] we have proved the theorem:

Theorem 1 *For any $\alpha < 2, k > 0$, there exists $\varepsilon^*(\alpha, k)$ such that for any $\varepsilon \leq \varepsilon^*(\alpha, k)$ and $z \leq \varepsilon^{-1/3}$ there exists*

$$\begin{aligned} \lim_{N,p \rightarrow \infty, \alpha_N \rightarrow \alpha} E\{f_{N,p}(k, h, z, \varepsilon)\} &= F(\alpha, k, h, z, \varepsilon), \\ F(\alpha, k, h, z, \varepsilon) &\equiv \max_{R>0} \min_{0 \leq q \leq R} \left[\alpha E \left\{ \log H \left(\frac{u\sqrt{q} + k}{\sqrt{\varepsilon + R - q}} \right) \right\} \right. \\ &\quad \left. + \frac{1}{2} \frac{q}{R - q} + \frac{1}{2} \log(R - q) - \frac{z}{2} R + \frac{h^2}{2} (R - q) \right], \end{aligned} \quad (1.8)$$

where u is a Gaussian random variable with zero mean and variance 1.

Similar results for small α were obtained in [T2] for the so-called Gardner-Derrida model. In the paper [T4] the fluctuation of the order parameters for the Gardner-Derrida model were studied, but only for small enough α .

An important ingredient of the analysis of the free energy of the model (1.5) in [S-T2] was the proof of the fact that the variance of its order parameters (or the overlap parameters) disappears in the thermodynamic limit. In the present paper we study the behaviour of fluctuations of the overlap parameters, defined as

$$R_{l,m} = \frac{1}{N} (\mathbf{J}^{(l)}, \mathbf{J}^{(m)}), \quad (l, m = 1, \dots, n), \quad (1.9)$$

where the upper indexes of the variables \mathbf{J} mean that we consider n replicas of the Hamiltonian (1.5) with the same random parameters $\{\boldsymbol{\xi}^{(\mu)}\}_{\mu=1}^p, \{\mathbf{h}\}$, but different $\mathbf{J}^{(1)}, \dots, \mathbf{J}^{(n)}$.

We introduce also the notations:

$$\begin{aligned} \dot{q} &= N^{1/2}(\langle R_{1,2} \rangle - q), \\ T_{l,m} &= \frac{1}{N^{1/2}}(\dot{\mathbf{J}}^{(l)}, \dot{\mathbf{J}}^{(m)}), \quad T_l = \frac{1}{N^{1/2}}(\dot{\mathbf{J}}^{(l)}, \langle \mathbf{J} \rangle). \end{aligned} \quad (1.10)$$

Here and below $\dot{\mathbf{J}} \equiv \mathbf{J} - \langle \mathbf{J} \rangle$ and (q, R) is the unique solution of the system of equations:

$$\begin{aligned} q &= (R - q)^2 \left[\frac{\alpha}{R - q + \varepsilon} E \left\{ A^2 \left(\frac{\sqrt{q}u + k}{\sqrt{R - q + \varepsilon}} \right) \right\} + h^2 \right], \\ z &= \frac{\alpha}{(R - q + \varepsilon)^{3/2}} E \left\{ (\sqrt{q}u + k) A \left(\frac{\sqrt{q}u + k}{\sqrt{R - q + \varepsilon}} \right) \right\} \\ &\quad - \frac{q}{(R - q)^2} + \frac{1}{R - q} + h^2, \end{aligned} \quad (1.11)$$

with

$$A(x) = -\frac{1}{\sqrt{2\pi}} \frac{d}{dx} \log H(x).$$

To avoid additional technical difficulties below we assume that $\{\xi_i^{(\mu)}\}$ are independent Gaussian random variables with zero mean and variance 1.

The main result of the paper is

Theorem 2 *Consider any $\alpha < 2$, $k > 0$, $\varepsilon \leq \varepsilon^*(\alpha, k)$ and $z \leq \varepsilon^{-1/3}$. Then for any integer n the families of random variables $\{\sqrt{N}(R_{l,m} - E\langle R_{l,m} \rangle)\}_{l < m \leq n}$, converges in distribution, as $N, p \rightarrow \infty, p/N \rightarrow \alpha$, to the Gaussian family of random variables $\{v_{l,m}\}_{l < m \leq n}$, with the covariance matrix:*

$$\begin{aligned} E\{v_{l,m}v_{l,m}\} &= A_0, \\ E\{v_{l,m}v_{l,m'}\} &= B_0 \quad (m \neq m'), \\ E\{v_{l,m}v_{l',m'}\} &= C_0 \quad (m, m', l, l' \text{ are different}). \end{aligned} \quad (1.12)$$

In particular,

$$\begin{aligned} \lim_{N, p \rightarrow \infty, p/N \rightarrow \alpha} E\{\langle T_{1,2}^{2n} \rangle\} &= \frac{\Gamma(2n-1)}{\Gamma(n-1)} A^n \\ \lim_{N, p \rightarrow \infty, p/N \rightarrow \alpha} E\{\langle T_1^{2n} \rangle\} &= \frac{\Gamma(2n-1)}{\Gamma(n-1)} B^n \\ \lim_{N, p \rightarrow \infty, p/N \rightarrow \alpha} E\{\dot{q}^{2n}\} &= \frac{\Gamma(2n-1)}{\Gamma(n-1)} C^n, \end{aligned} \quad (1.13)$$

where the constants A_0, B_0, C_0, A, B, C depend on $\alpha, k, z, \varepsilon$ and all odd moments for these random variables tend to zero.

Remark 1 *In fact it follows from our proof (see proofs of Lemmas 3,4,5 in Sec.2) that $\{T_{l,m}\}_{l < m \leq n}$ and $\{T_l\}_{l \leq n}$ in some sense do not depend on the random variables $\{\xi_i^{(\mu)}\}$ and $\{h_i\}$, i.e. if we consider P - some product of $\{T_{l,m}\}_{l < m \leq n}$ and $\{T_l\}_{l \leq n}$, then*

$$\lim_{N, p \rightarrow \infty, p/N \rightarrow \alpha} E\{(\langle P \rangle - E\langle P \rangle)^2\} = 0. \quad (1.14)$$

As it was mentioned above, similar results were obtained in [T4] for the Gardner-Derrida model for small α and for the Sherrington-Kirkpatrick model for the high temperature. We would like to mention also the work [Gu-T], where the fluctuations of the overlap parameters for the Sherrington-Kirkpatrick model in the high temperature region were studied by the method of characteristic functions.

One of the most important feature of our Hamiltonian (1.5), which allows us to prove Theorems 1 and 2 for any $\alpha < \alpha_c(k)$ is that it has the form

$$-\mathcal{H} = \sum_{\mu} g(S_{\mu}) - \frac{z}{2}(\mathbf{J}, \mathbf{J}) - (\mathbf{h}, \mathbf{J}), \quad S_{\mu} = \frac{1}{N^{1/2}}(\mathbf{J}, \boldsymbol{\xi}^{(\mu)}), \quad (1.15)$$

where $g(x)$ is a concave function. It allows us to use the Brascamp-Lieb inequalities (see [1]), according to which for any integer n and any $\mathbf{x} \in \mathbf{R}^N$

$$\left\langle \left(\frac{(\dot{\mathbf{J}}, \mathbf{x})}{\sqrt{N}} \right)^{2n} \right\rangle \leq \frac{\Gamma(2n-1)}{z^n \Gamma(n-1)} \left(\frac{|\mathbf{x}|^2}{N} \right)^n. \quad (1.16)$$

Besides, for any smooth function f

$$\langle (f - \langle f \rangle)^2 \rangle \leq \frac{1}{z} \langle |\nabla f|^2 \rangle. \quad (1.17)$$

Below we use the representation (1.15) of \mathcal{H} and the following properties of the functions $g(x)$:

$$g(x) \leq 0, \quad -C \leq g'' \leq 0, \quad |g^{(s)}(x)| \leq C, \quad (s = 3, \dots, 6). \quad (1.18)$$

In fact the only place where we use the real form of $g(x)$ is that the limiting system of equations (1.11) has the unique solution (see [S-T2]).

We use also notations:

$$S_\mu^{(l)} = \frac{1}{N^{1/2}}(\mathbf{J}^{(l)}, \boldsymbol{\xi}^{(\mu)}), \quad \tilde{R}_{l,l'} = \frac{1}{N} \sum_\mu g'(S_\mu^{(l)})g'(S_\mu^{(l')}), \quad \tilde{U}_l = \frac{1}{N} \sum_\mu (g''(S_\mu^{(l)}) + g'^2(S_\mu^{(l)})). \quad (1.19)$$

An important ingredient of our proof is the following proposition:

Proposition 1 *There exists $d_0 > 0$ such that for any $\delta < d_0$*

$$\begin{aligned} \text{Prob}\{|\langle R_{1,2} \rangle - E\langle R_{1,2} \rangle| > \delta\} &\leq e^{-N\delta^2/2C}, \\ \text{Prob}\{|\langle R_{1,1} \rangle - E\langle R_{1,1} \rangle| > \delta\} &\leq e^{-N\delta^2/2C}, \\ \text{Prob}\{|\langle \tilde{R}_{1,2} \rangle - E\langle \tilde{R}_{1,2} \rangle| > \delta\} &\leq e^{-N\delta^2/2C}, \\ \text{Prob}\{|\langle \tilde{U}_1 \rangle - E\langle \tilde{U}_1 \rangle| > \delta\} &\leq e^{-N\delta^2/2C}. \end{aligned} \quad (1.20)$$

Corollary 1

$$E\{|\langle R_{1,2} \rangle - E\langle R_{1,2} \rangle|^n\} \leq 2 \frac{C^n \Gamma(n)}{N^{n/2}}. \quad (1.21)$$

2 Proof of Main Results

Proof of Proposition 1

For the proof of Proposition 1 we need the following remark:

Remark 2 *It was proven in [S-T2] that there exist constants M_0, m_0 such that for any $M > M_0$*

$$\text{Prob}\{\langle (\mathbf{J}, \mathbf{J}) \rangle \geq MN\} \leq e^{-Nm_0(M-M_0)}. \quad (2.1)$$

Besides, it is well known that if we define

$$\mathcal{A}_{*i,j} = \frac{1}{N} \sum_\mu \xi_i^{(\mu)} \xi_j^{(\mu)}, \quad (2.2)$$

then there exist C_0, c_0 such that for any $C > C_0$

$$\text{Prob}\{\|\mathcal{A}_*\| \geq C\} \leq e^{-Nc_0(C-C_0)} \quad (2.3)$$

(see e.g. [S-T1]).

We prove Proposition 1 using a method proposed in [T4] with small modifications which we need to study the case, when the variables $\{J_i\}$ are unbounded.

Lemma 1 Consider two Gaussian independent random vectors $\{\mathbf{u}\}$ and $\{\mathbf{v}\}$. Let $f(\mathbf{x})$ satisfies the conditions:

$$P(A) = \text{Prob}\{|\nabla f(\mathbf{u})|^2 \geq A\} \leq e^{-C(A-A_0)} \quad (\forall A > A_0) \quad (2.4)$$

and for some s_0

$$E\{e^{\pm s_0 f}\} \leq e^{s_0 B}. \quad (2.5)$$

Then for $s \leq \frac{1}{4}s_0$

$$E\{e^{\pm s(f(\mathbf{u})-Ef(\mathbf{u}))}\} \leq e^{2A_0 s^2} (1 + (A_0 + C^{-1})e^{2sB-CA_0/2}). \quad (2.6)$$

Proof.

The proof is very simple. Consider

$$G_{s,t}(\mathbf{u}, \mathbf{v}) = \exp\{s(f(\mathbf{u}) - f(\sqrt{1-t}\mathbf{u} + \sqrt{t}\mathbf{v}))\}, \quad \varphi_s(t) = E\{G_{s,t}(\mathbf{u}, \mathbf{v})\}.$$

Then, integrating by parts, we get

$$\begin{aligned} \varphi'_s(t) &= \frac{s}{2} E \left\{ \frac{\partial}{\partial x_i} f(\sqrt{1-t}\mathbf{u} + \sqrt{t}\mathbf{v}) \left(\frac{u_i}{\sqrt{1-t}} - \frac{v_i}{\sqrt{t}} \right) G_{s,t}(\mathbf{u}, \mathbf{v}) \right\} \\ &= \frac{s^2}{2\sqrt{1-t}} E \left\{ \frac{\partial}{\partial x_i} f(\sqrt{1-t}\mathbf{u} + \sqrt{t}\mathbf{v}) \frac{\partial}{\partial x_i} f(\mathbf{u}) G_{s,t}(\mathbf{u}, \mathbf{v}) \right\} \\ &\leq \frac{A_0 s^2}{\sqrt{1-t}} \varphi_s(t) + \frac{s^2 e^{2sB}}{2\sqrt{1-t}} E^{1/2} \left\{ |\nabla f(u)|^4 \theta(|\nabla f(u)|^2 - 2A_0) \right\} \\ &\leq \frac{A_0 s^2}{\sqrt{1-t}} \varphi_s(t) + \frac{s^2 e^{2sB}}{2\sqrt{1-t}} \left(\int_{A>2A_0} A^2 dP(A) \right)^{1/2}. \end{aligned}$$

Thus we have got

$$E\{e^{s(f(\mathbf{u})-f(\mathbf{v}))}\} = \varphi_s(1) \leq e^{2A_0 s^2} (\varphi_s(0) + (A_0 + C^{-1})e^{2sB-CA_0/2}).$$

Since by definition $\varphi_s(0) = 1$, averaging first with respect to \mathbf{u} and then with respect to \mathbf{v} and using the Jensen inequality, we get (2.6). Lemma 1 is proven.

To apply this result to $f = N\langle R_{1,2} \rangle$ we have to check the condition (2.4). We write

$$\begin{aligned} \frac{1}{N} \sum \left| \frac{\partial f}{\partial \xi_i^{(\mu)}} \right|^2 &= \frac{4}{N^2} \sum_{i,j,k,\mu} \langle J_i \rangle \left\langle \dot{J}_i g'(S_\mu) J_j \right\rangle \left\langle J_j g'(S_\mu) \dot{J}_k \right\rangle \langle J_k \rangle \\ &\leq \frac{8}{N^2} \sum_{i,j,k,\mu} \langle J_i \rangle \left\langle \dot{J}_i g'(S_\mu) \dot{J}_j \right\rangle \left\langle \dot{J}_j g'(S_\mu) \dot{J}_k \right\rangle \langle J_k \rangle \\ &\quad + \frac{8}{N^2} \sum_{i,j,k,\mu} \langle J_i \rangle \left\langle \dot{J}_i (g'(S_\mu) - \langle g'(S_\mu) \rangle) \right\rangle \left\langle (g'(S_\mu) - \langle g'(S_\mu) \rangle) \dot{J}_k \right\rangle \langle J_k \rangle \langle J_j \rangle^2 \\ &= 8I + 8II. \end{aligned} \quad (2.7)$$

To estimate the r.h.s. we use the following proposition:

Proposition 2 Consider the matrices $\mathcal{A}^{(f)} : \mathbf{R}^N \rightarrow \mathbf{R}^N$, $\mathcal{B}^{(f)} : \mathbf{R}^p \rightarrow \mathbf{R}^N$ and $\mathcal{C} : \mathbf{R}^p \rightarrow \mathbf{R}^p$ of the form

$$\begin{aligned} \mathcal{A}_{i,j}^{(f)} &= \left\langle \dot{J}_i \dot{J}_j f(\mathbf{J}) \right\rangle, \quad \mathcal{B}_{i,\mu}^{(f)} = \left\langle \dot{J}_i (g'(S_\mu) - \langle g'(S_\mu) \rangle) f(\mathbf{J}) \right\rangle, \\ \mathcal{C}_{\mu,\nu} &= \left\langle (g'(S_\mu) - \langle g'(S_\mu) \rangle) (g'(S_\nu) - \langle g'(S_\nu) \rangle) \right\rangle. \end{aligned} \quad (2.8)$$

Then

$$\|\mathcal{A}^{(f)}\| \leq \frac{\langle 3f^2 \rangle^{1/2}}{z}, \quad \|\mathcal{B}^{(f)}\| \leq \frac{\|\mathcal{A}_*\|^{1/2} \langle |g''|^2 \rangle^{1/2} \langle 3f^4 \rangle^{1/4}}{z}, \quad \|\mathcal{C}\| \leq \frac{\|\mathcal{A}_*\| \langle |g''|^2 \rangle}{z}, \quad (2.9)$$

where the matrix \mathcal{A}_* is defined by (2.2).

We prove this proposition in the next section.

Denoting $\mathcal{A}_{i,j}^\mu = \langle \dot{J}_i \dot{J}_j g'(S_\mu) \rangle$ and using (2.9), we obtain

$$I = \frac{1}{N^2} \sum (\mathcal{A}^\mu * \mathcal{A}^\mu \langle \mathbf{J} \rangle, \langle \mathbf{J} \rangle) \leq \frac{3}{z^2 N^2} (\langle \mathbf{J} \rangle, \langle \mathbf{J} \rangle) \left\langle \sum g'^2(S_\mu) \right\rangle.$$

Similarly, taking $f = 1$ in the definition (2.9) of \mathcal{B} , we have got

$$II = \frac{1}{N^2} (\langle \mathbf{J} \rangle, \langle \mathbf{J} \rangle) (\mathcal{B} * \mathcal{B}^* \langle \mathbf{J} \rangle, \langle \mathbf{J} \rangle) \leq \frac{2 \|\mathcal{A}_*\| \langle |g''|^2 \rangle}{z^2 N^2} (\langle \mathbf{J} \rangle, \langle \mathbf{J} \rangle)^2.$$

Proposition 3

$$\frac{1}{N} \sum \langle g'^2(S_\mu) \rangle \leq \frac{C}{N^2} \sum (\boldsymbol{\xi}^{(\mu)}, \boldsymbol{\xi}^{(\mu)}).$$

Since g'' is bounded function, by using this proposition (see the next section for the proof) one can easily check the condition (2.4) for the terms I and II .

Thus we have proved the first line of (2.6). Now by using the standard Chebyshev inequality we get (1.20). The other inequalities in (1.20) can be proven similarly.

To prove Theorem 2 we need to make some preliminary work.

Denote

$$d = q(R - q)^{-1}, \quad U = d + z - (R - q)^{-1}; \quad (2.10)$$

Lemma 2 *For any $0 < \epsilon < 1$ there exists a constant C_ϵ such that uniformly in N*

$$\begin{aligned} |E\{\langle R_{1,2} \rangle\} - q| &\leq \frac{C_\epsilon}{N^{1-\epsilon}}, & |E\{\langle R_{1,1} \rangle\} - R| &\leq \frac{C_\epsilon}{N^{1-\epsilon}}, \\ |E\{\langle g'(S_\mu) \rangle^2\} - d| &\leq \frac{C_\epsilon}{N^{1-\epsilon}}, & |E\{\langle g''(S_\mu) + g'^2(S_\mu) \rangle\} - U| &\leq \frac{C_\epsilon}{N^{1-\epsilon}}. \end{aligned} \quad (2.11)$$

For the proof of this lemma see Section 3.

Using this lemma and inequality (1.21), we get

$$E\{|q^n|\} \leq 2C^n \Gamma(n). \quad (2.12)$$

Besides, using inequalities (1.16) one can get easily:

$$E\{\langle T_{1,2}^n \rangle\} \leq C^n \Gamma(n), \quad E\{\langle T_1^n \rangle\} \leq C^n \Gamma(n).$$

Then, since $R_{1,2} - q = N^{-1/2}(T_{1,2} + T_1 + T_2 + \dot{q})$, we obtain

$$E\{|\langle (R_{1,2} - q)^n \rangle|\} \leq \frac{C^n \Gamma(n)}{N^{n/2}}. \quad (2.13)$$

Besides, on the basis of (1.17) and Lemma 2, we have got

$$E\{\langle (\tilde{R}_{l,l'} - d)^2 \rangle\} \leq \frac{C}{N}, \quad E\{\langle \dot{U}_l^2 \rangle\} \leq \frac{C}{N}, \quad E\{\langle (R_{l,l} - R)^2 \rangle\} \leq \frac{C}{N}. \quad (2.14)$$

Here and below we denote

$$\dot{U}_l = \tilde{U}_l - U.$$

From this inequality, using the bound

$$|\tilde{R}_{l,l'}|, |\dot{U}^2| \leq N^{-1} \langle (\mathbf{J}, \mathbf{J}) \rangle \|\mathcal{A}_*\|$$

and inequalities (2.1) and (2.3), we obtain for any $r > 2$

$$E \left\{ \langle |\tilde{R}_{l,l'} - d|^r \rangle \right\} \leq \frac{C_r}{N}, \quad E \left\{ \langle |\dot{U}|^r \rangle \right\} \leq \frac{C_r}{N}, \quad E \left\{ \langle |R_{l,l} - R|^r \rangle \right\} \leq \frac{C_r}{N}. \quad (2.15)$$

Following the method of [T5], we introduce the Hamiltonian

$$\begin{aligned} -H_t = & \sum g(S_\mu^- + J_1 \sqrt{t} \xi_1^{(\mu)} N^{-1/2}) + \sqrt{d(1-t)} u J_1 \\ & + \frac{1-t}{2} (U-d) J_1^2 - \frac{z}{2} J_1^2 - \frac{z}{2} (\mathbf{J}^-, \mathbf{J}^-), \end{aligned} \quad (2.16)$$

where u is a normally distributed random variable, independent of $\xi^{(\mu)}$ and \mathbf{h} and $S_\mu^- = N^{-1/2}(\xi^{(\mu)}, \mathbf{J}^-)$ do not depend on $\xi_1^{(\mu)}$.

Denote $\langle \dots \rangle_t$ the Gibbs averaging to H_t (or n replicas of H_t), and for any $\xi_1^{(\mu)}$ -independent function defined of $\mathbf{R}^{N \times n}$ define

$$\nu_t(f) = E \langle f \rangle_t, \quad \nu'_t(f) = \frac{d}{dt} \nu_t(f). \quad (2.17)$$

for any $\xi_1^{(\mu)}$ -independent function defined of $\mathbf{R}^{N \times n}$. Besides, to simplify notation we denote

$$s_i = J_1^{(i)}.$$

Proposition 4 *For any integer n*

$$|\nu_t(s_1^{2n})| \leq C^n \Gamma(n). \quad (2.18)$$

For the proof of this proposition see the next section.

Let us compute $\nu'_t(f)$. Differentiating and then integrating by parts with respect to $\xi_1^{(\mu)}$ and u , we have got

$$\begin{aligned} \nu'_t(f) = & \frac{1}{2} \sum_{l=1}^n \nu_t(f s_l^2 \dot{U}_l^-) - \frac{n}{2} \nu_t(f s_{n+1}^2 \dot{U}_{n+1}^-) \\ & + \sum_{l < l'}^n \nu_t(f s_l s_{l'} (\tilde{R}_{l,l'}^- - d)) - n \sum_{l=1}^n \nu_t(f s_l s_{n+1} (\tilde{R}_{l,n+1}^- - d)) \\ & + \frac{n(n+1)}{2} \nu_t(f s_{n+1} s_{n+2} (\tilde{R}_{n+1,n+2}^- - d)). \end{aligned} \quad (2.19)$$

Since the Hamiltonian (2.16) has the form (1.15), the inequalities (1.16) and (1.17) for this Hamiltonian are also valid. Therefore the estimate (2.13), and (2.15) are fulfilled and so, using the Schwartz inequality and (2.15), we get

$$|\nu_1(f) - \nu_0(f)| \leq C \max_t \nu_t^{1/2}(|f|^2) N^{-1/2}. \quad (2.20)$$

Using the same formula to compute the second derivative of $\nu_t(f)$ with respect to t , we obtain the expression in each term of which we have $(\tilde{R}_{l,l'} - d)(\tilde{R}_{l_1,l'_1} - d)$ or $(\tilde{R}_{l,l'} - d)\dot{U}_{l_1}$ or $\dot{U}_l \dot{U}_{l_1}$. Using again (2.15) and the Hölder inequality, we obtain for any $0 < \epsilon < 1$:

$$|\nu_1(f) - \nu_0(f) - \nu'_0(f)| \leq \max_t \nu_t^{\epsilon/2}(|f|^{2/\epsilon}) \nu_t^{\epsilon/2}(|s_1|^{4/\epsilon}) N^{-1+\epsilon}. \quad (2.21)$$

To compute the averages of the type $\langle \tilde{R}_{l,l'} \rangle$ we use another tool. Denote $\mathcal{D}_l = \frac{d}{dS_1^{(l)}}$. One can see easily that, e.g., $E\{\langle \tilde{R}_{1,2} \rangle\}$ can be represented in the form

$$E\{\langle \tilde{R}_{1,2} \rangle\} = E \left\{ \frac{\langle \mathcal{D}_1 \mathcal{D}_2 G(S_1^{(1)}) G(S_1^{(2)}) \rangle_{(p-1)}}{\langle G(S_1^{(1)}) G(S_1^{(2)}) \rangle_{(p-1)}} \right\},$$

where $G(S) = e^{g(S)}$ and the symbol $\langle \dots \rangle_{(p-1)}$ means the averaging with respect to the Hamiltonian (1.15) in which $g(S_1)$ is replaced by 0.

Let us again consider a standard Gaussian variable u and introduce a function

$$G_t(S, u) = \frac{1}{\sqrt{(1-t)(R-q+\varepsilon)}} \int e^{-x^2/2(R-q+\varepsilon)(1-t)} G(\sqrt{t}S + u\sqrt{q(1-t)} + x) dx. \quad (2.22)$$

In particular,

$$G_0(u) = \frac{1}{\sqrt{(R-q+\varepsilon)}} \int e^{-x^2/2(R-q+\varepsilon)} G(u\sqrt{q} + x) dx. \quad (2.23)$$

It is evident, that $G_0(S, u) = G_0(u)$ (i.e. it does not depend on S) and $G_1(S, u) = G(S)$. We remark, that the definition (2.22) becomes more natural, if we introduce it through the Fourier transform $\hat{G}(\lambda)$ of the function $G(S)$:

$$G_t(S, u) = \frac{1}{\sqrt{2\pi}} \int \hat{G}(\lambda) \exp\{-i\lambda(S\sqrt{t} + u\sqrt{q(1-t)}) - \frac{1-t}{2}(R-q+\varepsilon)\lambda^2\}. \quad (2.24)$$

Now for any $\xi_i^{(1)}$ -independent function $f : \mathbf{R}^{N \times n} \rightarrow \mathbf{R}$ and some polynomial $P(x_1, \dots, x_n)$ consider the operator $\mathcal{P}_t = P(t^{-1/2}\mathcal{D}_1, \dots, t^{-1/2}\mathcal{D}_n)$

$$\begin{aligned} \varphi_t^{(n)}(f\mathcal{P}_t) &= E \left\{ \frac{\langle f \mathcal{P}_t G_t(S_1^{(1)}, u) \dots G_t(S_1^{(n)}, u) \rangle_{(p-1)}}{\langle G_t(S_1^{(1)}, u) \dots G_t(S_1^{(n)}, u) \rangle_{(p-1)}} \right\} \\ &= E \left\{ \left\langle f \frac{\mathcal{P}_t G_t(S_1^{(1)}, u) \dots G_t(S_1^{(n)}, u)}{G_t(S_1^{(1)}, u) \dots G_t(S_1^{(n)}, u)} \right\rangle_{(t)} \right\}, \end{aligned} \quad (2.25)$$

where $\langle \dots \rangle_{(t)}$ means the Gibbs averaging corresponding to the n replicas of the Hamiltonian (1.15) in which $g(S_1)$ is substituted by $\log G_t(S_1, u)$. According to the result of [1], this function is also concave with respect to S_1 and so inequalities (2.13) and (2.14) for it are also valid.

We remark here also that due to the definition G_t (2.22) the operator \mathcal{P}_t has a natural form:

$$\begin{aligned} \mathcal{P}_t G_t(S_1^{(1)}, u) \dots G_t(S_1^{(n)}, u) &= \frac{1}{(\sqrt{2\pi})^n} \int \hat{G}(\lambda_1) \dots \hat{G}(\lambda_n) P(-i\lambda_1, \dots, -i\lambda_n) \\ &\quad \exp\{-i \sum \lambda_l S_1^{(l)} \sqrt{t} - iu \sum \lambda_l \sqrt{q(1-t)} - \frac{1-t}{2}(R-q)\lambda^2\}. \end{aligned} \quad (2.26)$$

So for $t = 0$ it is well defined:

$$\mathcal{P}_t G_t(S_1^{(1)}, u) \dots G_t(S_1^{(n)}, u) \Big|_{t=0} = P(q^{-1/2} \frac{d}{dx_1}, \dots, q^{-1/2} \frac{d}{dx_1}) G_0(x_1) \dots G_0(x_n) \Big|_{x_1=\dots=x_n=u}.$$

Let us compute the derivative with respect to t of $\varphi_t^{(n)}(f\mathcal{P}_t)$.

$$\begin{aligned} \frac{d}{dt} \varphi_t^{(n)}(f\mathcal{P}_t) &= \frac{1}{2} \sum_{l=1}^n \varphi_t^{(n)}(f(R_{l,l} - R)t^{-1}\mathcal{D}_l^2 \mathcal{P}_t) - \frac{n}{2} \varphi_t^{(n+1)}(f(R_{n+1,n+1} - R)t^{-1}\mathcal{D}_{n+1}^2 \mathcal{P}_t) \\ &\quad + \sum_{l < l'}^n \varphi_t^{(n)}(f(R_{l,l'} - q)t^{-1}\mathcal{D}_l \mathcal{D}_{l'} \mathcal{P}_t) - n \sum_{l=1}^n \varphi_t^{(n)}(f(R_{l,l} - q)t^{-1}\mathcal{D}_l \mathcal{D}_{n+1} \mathcal{P}_t) \\ &\quad + \frac{n(n+1)}{2} \varphi_t^{(n+2)}(f(R_{n+1,n+2} - q)t^{-1}\mathcal{D}_{n+1} \mathcal{D}_{n+2} \mathcal{P}_t). \end{aligned} \quad (2.27)$$

This formula can be obtained easily if we differentiate with respect to t and then integrate by parts with respect to $\xi_i^{(1)}$ and u in the expressions (2.24) and (2.26).

Proposition 5 For any polynomial $P(\lambda_1, \dots, \lambda_n)$ ($\deg P \leq 6$)

$$|\varphi_t^{(n)}(\mathcal{P}_t)| \leq C, \quad (2.28)$$

where constant C depends on n and on the polynomial $P(x_1, \dots, x_n)$.

Remark 3 If we take $g(x) = -\log H(\frac{x}{\sqrt{\varepsilon}})$, then Proposition 5 is valid for the polynomial of any degree.

As it was mentioned above the inequalities (1.16) and (1.17) for $\langle \dots \rangle_{(t)}$ are also valid. Therefore the estimate (2.13), and (2.15) are fulfilled and so, using the Schwartz inequality, Proposition 5 and (2.15), we obtain:

$$|\varphi_1^{(n)}(f\mathcal{P}_1) - \varphi_0^{(n)}(f\mathcal{P}_0)| \leq C \max_t \nu_t^{1/4} (|f|^4) N^{-1/2}. \quad (2.29)$$

Using the same formula to compute the second derivative of $\varphi_t^{(n)}(f\mathcal{P}_t)$ with respect to t , we obtain the expression in each term of which we have $(R_{l,l'} - q)(R_{l_1,l'_1} - q)$ or $(\tilde{R}_{l,l'} - q)(R_{l_1,l_1} - R)$ or $(R_{l_1,l_1} - R)^2$. Using the Hölder inequality, Proposition 5 and (2.15), we obtain:

$$|\varphi_1^{(n)}(f\mathcal{P}_1) - \varphi_0^{(n)}(f\mathcal{P}_0) - \frac{d}{dt} \varphi_0^{(n)}(f\mathcal{P}_0)| \leq C \max_t \nu_t^{1/4} (|f|^4) N^{-1}. \quad (2.30)$$

Proof of Theorem 2

We prove Theorem 2 in 3 steps which are Lemma 3, 4 and 5.

Lemma 3 Consider an expression of the form $T_{1,2}^k P$ where P is some product of the terms $T_{i,j}$ (different from $T_{1,2}$), T_i and \dot{q} . Then

$$E\langle T_{1,2}^k P \rangle = (k-1)AE\langle T_{1,2}^{k-2} P \rangle + O(N^{-1/2+\epsilon}), \quad (2.31)$$

where

$$A = \frac{b_0}{1 - \alpha b_0 c_0}, \quad (2.32)$$

with

$$b_0 = (R - q)^2, \quad c_0 = q^{-2} E\{(g_0''(u))^2\}, \quad g_0(u) = \log G_0(u), \quad (2.33)$$

where $E\{..\}$ means the averaging with respect to the standard Gaussian random variable u and $G_0(u)$ was defined in (2.23).

Lemma 4 Consider an expression of the form $T_1^k P$ where P is some product of the terms T_i with $i \neq 1$ and \dot{q} . Then

$$E\langle T_1^k P \rangle = (k-1)BE\langle T_1^{k-2} P \rangle + O(N^{-1/2+\epsilon}) \quad (2.34)$$

where B is some N -independent constant which is an algebraic expression of the coefficients b_0 , c_0 and

$$\begin{aligned} b_1 &= E\{\langle s^2 \rangle_0 \langle s \rangle_0^2\} = q(R - q) \\ c_1 &= E\{\langle (\mathcal{D}_1 - \mathcal{D}_2) \mathcal{D}_1 \mathcal{D}_3 \mathcal{D}_4 \rangle_{(0)}\} = q^{-2} E\{g_0''(g_0')^3\} \\ c_2 &= E\{\langle (\mathcal{D}_1 - \mathcal{D}_2) \mathcal{D}_1^2 \mathcal{D}_3 \rangle_{(0)}\} = q^{-2} E\{(g_0''' + 2g_0''g_0')g_0'\} \\ c_3 &= E\{\langle (\mathcal{D}_1^2 - \mathcal{D}_2^2) \mathcal{D}_1^2 \rangle_{(0)}\} = q^{-2} E\{(g_0^{(4)} + 4g_0'''g_0' + 2(g_0'')^2 + 4g_0''(g_0')^2)\} \\ c_4 &= E\{\langle \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4 \rangle_{(0)}\} = q^{-2} E\{(g_0')^4\}. \end{aligned} \quad (2.35)$$

For $h = 0$

$$B = 2 \frac{\partial^2 F}{\partial z^2} + \frac{1}{2} A \quad (2.36)$$

Lemma 5

$$E\langle \dot{q}^k \rangle = (k-1)CE\langle \dot{q}^{k-2} \rangle, \quad (2.37)$$

where C is some N -independent constant which is an algebraic expression of the coefficients $b_{0,1}, c_{0,1,2,3,4}$.

One can see easily that the statement of Theorem 2 follows from these lemmas by induction. So our goal is to prove the Lemmas.

Proof of Lemma 3. First of all we rewrite all terms of our initial product (including all $T_{1,2}$) in the form

$$\begin{aligned} T_{l,l'} &= N^{-1/2}(\mathbf{J}^{-(l)} - \mathbf{J}^{-(k)})(\mathbf{J}^{-(l)} - \mathbf{J}^{-(k')}) + N^{-1/2}(J_1^{(l)} - J_1^{(k)})(J_1^{(l)} - J_1^{(k')}) \\ T_l &= N^{-1/2}(\mathbf{J}^{-(l)} - \mathbf{J}^{-(k_1)})\mathbf{J}^{-(k'_1)} + N^{-1/2}(J_1^{(l)} - J_1^{(k_1)})J_1^{(k'_1)} \\ \dot{q} &= N^{-1/2}((\mathbf{J}^{-(k_2)}, \mathbf{J}^{-(k'_2)}) - (N-1)q) + N^{-1/2}(J_1^{(k_2)}J_1^{(k'_2)} - q), \end{aligned} \quad (2.38)$$

where indexes $k, k', k_1, k'_1, k_2, k'_2$ are different for each term in the product and different from initial indexes l, l' . We denote the last term in i -th expression by $N^{-1/2}f_i(J_1)$. Using the symmetry of the Hamiltonian and the above representation, we can write

$$\begin{aligned} E\{T_{1,2}^k P\} &= \sqrt{N}\nu_1((s_1 - s_k)(s_2 - s_{k'})\tilde{P}^-) \\ &\quad + \sum_i \nu_1((s_1 - s_k)(s_2 - s_{k'})f_i(s)\tilde{P}_i^-) + O(N^{-1/2}) \\ &= I + II + O(N^{-1/2}), \end{aligned} \quad (2.39)$$

where \tilde{P}^- means the product only of such terms of (2.38) which does not contain s_l (including that, corresponding to $T_{1,2}$) and \tilde{P}_i^- means the product of the same terms except the i -th one. The term $O(N^{-1/2})$ appears because of the products which contain more than 1 term $f_i(s)$. Applying formula (2.20) to the term II , we have got

$$\begin{aligned} II &= \sum_i \nu_0((s_1 - s_k)(s_2 - s_{k'})f_i(s))\nu_1(\tilde{P}_i^-) + O(N^{-1/2}) \\ &= \sum_i \nu_0((s_1 - s_k)(s_2 - s_{k'})f_i(s))\nu_1(\tilde{P}_i) + O(N^{-1/2}). \end{aligned}$$

But, if $f_i(s)$ does not contain both s_1 and s_2 ,

$$\nu_0((s_1 - s_k)(s_2 - s_{k'})f_i(s)) = 0.$$

So we obtain

$$II = (k-1)b_0E\{\langle T_{1,2}^{k-2} P \rangle\} + O(N^{-1/2}). \quad (2.40)$$

Now let us analyze term I , using formula (2.21). It is evident that ν_0 term here is equal to 0. Calculating ν'_0 , we get

$$I = \sqrt{N} \sum_{l < l'}^n \nu_0((s_1 - s_k)(s_2 - s_{k'})s_l s_{l'})\nu_1(\tilde{R}_{l,l'}^- - d)\tilde{P}^-) + O(N^{-1/2+\epsilon}).$$

All the rest terms in (2.19) disappear because

$$\nu_0((s_1 - s_k)(s_2 - s_{k'})s_l s_{l'}) = b_0(\delta_{l,1}\delta_{l',2} + \delta_{l,k}\delta_{l',k'} - \delta_{l,k}\delta_{l',2} - \delta_{l,k'}\delta_{l',1}).$$

So, we have got

$$\begin{aligned} I &= b_0 \sqrt{N} E\{\langle (\tilde{R}_{1,2} - \tilde{R}_{1,k'} - \tilde{R}_{2,k} + \tilde{R}_{k,k'}) \tilde{P} \rangle\} + O(N^{-1/2+\epsilon}) \\ &= b_0 III + O(N^{-1/2+\epsilon}). \end{aligned} \quad (2.41)$$

To analyze III we use again the symmetry of (1.15) and notations of (2.25) to write

$$\begin{aligned} \alpha^{-1} III &= \sqrt{N} E\{\langle (g'(S_1^{(1)}) - g'(S_1^{(k)}))(g'(S_1^{(2)}) - g'(S_1^{(k')})) \tilde{P} \rangle\} \\ &= N^{1/2} \varphi_1((\mathcal{D}_1 - \mathcal{D}_k)(\mathcal{D}_2 - \mathcal{D}_{k'}) \tilde{P}). \end{aligned}$$

Now, applying formula (2.30), we can write

$$\alpha^{-1} III = \sqrt{N} \sum_{l < l'} \phi_0((\mathcal{D}_1 - \mathcal{D}_k)(\mathcal{D}_2 - \mathcal{D}_{k'}) \mathcal{D}_l \mathcal{D}_{l'}) E\{\langle (R_{l,l'} - q) \tilde{P} \rangle\} + O(N^{-1/2+\epsilon}). \quad (2.42)$$

All the rest terms in (2.27) disappear because

$$\varphi_0((\mathcal{D}_1 - \mathcal{D}_k)(\mathcal{D}_2 - \mathcal{D}_{k'})) = 0, \quad \varphi_0((\mathcal{D}_1 - \mathcal{D}_k)(\mathcal{D}_2 - \mathcal{D}_{k'}) \mathcal{D}_l^2) = 0.$$

Let us remark that

$$\varphi_0((\mathcal{D}_1 - \mathcal{D}_k)(\mathcal{D}_2 - \mathcal{D}_{k'}) \mathcal{D}_l \mathcal{D}_{l'}) = c_0(\delta_{l,1} \delta_{l',2} + \delta_{l,k} \delta_{l',k'} - \delta_{l,k} \delta_{l',2} - \delta_{l,k'} \delta_{l',1}),$$

with c_0 defined in (2.33).

Hence we get from (2.42) that

$$\begin{aligned} \alpha^{-1} III &= c_0 \sqrt{N} E\{\langle (R_{1,2} - R_{1,k'} - R_{2,k} + R_{k,k'}) \tilde{P} \rangle\} + O(N^{-1/2+\epsilon}) \\ &= c_0 E\{\langle T_{1,2}^k P \rangle\} + O(N^{-1/2+\epsilon}). \end{aligned} \quad (2.43)$$

Now, using relations (2.38), (2.39), (2.40), (2.41), (2.42) and (2.43), we get (2.31). Lemma 3 is proven.

Proof of Lemma 4

Like in the proof of Lemma 2 we use representation (2.38) for all the terms and let for the first T_1 the numbers of replicas here are 1, 2, 3 and for the i -th T_1 they are $(1, l_i, l_i + 1)$. Using the symmetry of the problem similarly to (2.39) write

$$\begin{aligned} E\{T_1^k P\} &= \sqrt{N} \nu_1((s_1 - s_2) s_3 \tilde{P}^-) \\ &\quad + \sum_{i=2}^k \nu_1((s_1 - s_2) s_3 (s_1 - s_{k_i}) s_{k'_i} f_i(s) \tilde{P}_i^-) + O(N^{-1/2}) \\ &= I + II + O(N^{-1/2}), \end{aligned} \quad (2.44)$$

where \tilde{P}^- means the product only of such terms of (2.38) which does not contain s_1 and \tilde{P}_i^- means the product of the same terms except the i -th one. By the same way as in Lemma 2 we get

$$II = (k-1) b_1 E\{T_1^{k-2} P\} + O(N^{-1/2}), \quad (2.45)$$

where b_1 is defined by (2.35). Using the formula (2.21) we have got

$$\begin{aligned} \alpha^{-1} I &= \sqrt{N} \left[2b_1 E\{\langle (\tilde{U}_1 - \tilde{U}_2) \tilde{P} \rangle\} + b_0 E\{\langle (\tilde{R}_{1,3} - \tilde{R}_{2,3}) \tilde{P} \rangle\} \right. \\ &\quad \left. + b_1 \sum_{l=3}^n E\{\langle (\tilde{R}_{1,l} - \tilde{R}_{2,l}) \tilde{P} \rangle\} - n b_1 E\{\langle (\tilde{R}_{1,n+1} - \tilde{R}_{2,n+1}) \tilde{P} \rangle\} \right] + O(N^{-1/2+\epsilon}). \end{aligned} \quad (2.46)$$

Using again the symmetry and notations (2.25), we write

$$\begin{aligned}\alpha^{-1}I &= \sqrt{N} \left[2b_1\varphi_1((\mathcal{D}_1^2 - \mathcal{D}_2^2)\tilde{P}) + b_0\varphi_1((\mathcal{D}_1 - \mathcal{D}_2)\mathcal{D}_3\tilde{P}) \right. \\ &\quad \left. + b_1 \sum_{l=3}^n \varphi_1((\mathcal{D}_1 - \mathcal{D}_2)\mathcal{D}_l\tilde{P}) - nb_1\varphi_1(\mathcal{D}_1 - \mathcal{D}_2)\mathcal{D}_{n+1}\tilde{P} \right] + O(N^{-1/2+\epsilon}).\end{aligned}$$

Applying formula (2.27) we get

$$\begin{aligned}\alpha^{-1}I &= \sqrt{N} \left[a_1^{(1)} E\{\langle (R_{1,1} - R_{2,2})\tilde{P} \rangle\} + a_2^{(1)} E\{\langle (R_{1,3} - R_{2,3})\tilde{P} \rangle\} \right. \\ &\quad \left. + a_3^{(1)} \sum_{l=3}^n E\{\langle (R_{1,l} - R_{2,l} - R_{1,n+1} + R_{2,n+1})\tilde{P} \rangle\} \right] + O(N^{-1/2+\epsilon}) \\ &= a_1^{(1)} E\{\langle (R_{1,1} - R_{2,2})\tilde{P} \rangle\} \sqrt{N} + a_2^{(1)} E\{T_1^k P\} \\ &\quad + (k-1)Aa_3^{(1)} E\{T_1^{k-2}P\} + O(N^{-1/2+\epsilon}),\end{aligned}\tag{2.47}$$

where $a_{1,2,3}^{(1)}$ are some algebraic combinations of $b_{0,1}$ and $c_{0,1,2,3,4}$ defined by (2.35).

Here we have used Lemma 3, according to which

$$\sqrt{N}E\{\langle (R_{1,l} - R_{2,l} - R_{1,n+1} + R_{2,n+1})\tilde{P} \rangle\} = E\{\langle T_{1,l}^2 T^{k-2}P \rangle\} = AE\{T_1^{k-2}P\} + O(N^{-1/2+\epsilon})$$

for $l = l_i$ ($i = 2, \dots, k$) and it is zero for the rest of l . Now we are faced with a problem to compute $E\{\langle (R_{1,1} - R_{2,2})\tilde{P} \rangle\} \sqrt{N}$.

By using the same procedure we can get

$$\begin{aligned}\sqrt{N}E\{\langle (R_{1,1} - R_{2,2})T_1^{k-1}P \rangle\} &= 2b_1(k-1)E\{T_1^{k-2}P\} \\ &\quad + \alpha a_1^{(2)} E\{\langle (R_{1,1} - R_{2,2})T_1^{k-1}P \rangle\} \sqrt{N} \\ &\quad + \alpha a_2^{(2)} E\{T_1^k P\} + (k-1)\alpha a_3^{(2)} E\{T_1^{k-2}P\} + O(N^{-1/2+\epsilon}),\end{aligned}\tag{2.48}$$

where $a_{1,2,3}^{(2)}$ are some algebraic combinations of $b_{0,1}$ and $c_{0,1,2,3,4}$ defined by (2.35).

Now we have got the system of two equations with respect to $E\{T_1^k P\}$ and

$E\{\langle (R_{1,1} - R_{2,2})T_1^{k-1}P \rangle\} \sqrt{N}$. This system gives us

$$E\{T_1^k P\} = (k-1)BE\{T_1^{k-2}P\}, \quad E\{\langle (R_{1,1} - R_{2,2})T_1^{k-1}P \rangle\} \sqrt{N} = (k-1)\tilde{B}E\{T_1^{k-2}P\} \sqrt{N} \tag{2.49}$$

with some B and \tilde{B} . But using the fact that B and \tilde{B} do not depend on k , we observe that

$$\begin{aligned}2 \frac{\partial^2 F}{\partial z \partial h} &= N^{-1} \sum_{i,j} E\{\langle J_i^2 J_j \rangle \tilde{h}_j\} = 4 \frac{\partial^2 F}{\partial z^2} - 2E\{\langle R_{1,1} T_1 \rangle\} \\ &= 4 \frac{\partial^2 F}{\partial z^2} - 2\tilde{B}.\end{aligned}$$

Then since $\frac{\partial^2 F}{\partial z \partial h} = 0$ for $h = 0$, we get

$$\tilde{B} = 2 \frac{\partial^2 F}{\partial z^2}.\tag{2.50}$$

Similarly one can get

$$\begin{aligned}\frac{\partial^2 F}{\partial h^2} &= N^{-1} \sum_{i,j} E\{\tilde{h}_i \langle J_i J_j \rangle \tilde{h}_j\} = R - q + 2 \frac{\partial^2 F}{\partial z \partial h} - 4\tilde{B} + 4E\{\langle T_1^2 \rangle\} - 2E\{\langle T_{1,2}^2 \rangle\} \\ &= R - q - 4\tilde{B} + 4B - 2A.\end{aligned}$$

Thus, using the fact that $\frac{\partial^2 F}{\partial h^2} = R - q$ for $h = 0$, we have got formula (2.36) for B .

Proof of Lemma 5

We write

$$\begin{aligned} E\{\dot{q}^k\} &= \sqrt{N}\nu_1((s_1 s_2 - q)\tilde{P}^-) \\ &\quad + \sum_{i=2}^k \nu_1((s_1 s_2 - q)^2 \tilde{P}_i^-) + O(N^{-1/2}) \\ &= I + II + O(N^{-1/2}), \end{aligned} \quad (2.51)$$

where

$$\tilde{P}^- = \prod_{l=2}^k N^{1/2} (R_{2l-1, 2l} - q)$$

One can see easily

$$II = (k-1)q^2 E\{\dot{q}^{k-2}\} + O(N^{-1/2}) \quad (2.52)$$

Calculating I with (2.20) we get

$$\begin{aligned} I &= \sqrt{N} \left[q^2 \sum_{l < l'} \nu_1((\tilde{R}_{l, l'} - \tilde{R}_{l, n+1} - \tilde{R}_{l', n+2} + \tilde{R}_{n+1, n+2})\tilde{P}^-) \right. \\ &\quad - (2b_1 - q^2) \sum_{l \geq 3} \nu_1((\tilde{R}_{1, 2} - \tilde{R}_{l, 2})\tilde{P}^-) - \frac{q^2}{2} \sum_{l \geq 3} \nu_1((\tilde{R}_{1, 1} - \tilde{R}_{l, l})\tilde{P}^-) \\ &\quad \left. + (b_0 - 2b_1)\nu_1((\tilde{R}_{1, 2} - d)\tilde{P}^-) + 2b_1\nu_1((\tilde{R}_{1, 1} - U)\tilde{P}^-) \right] + O(N^{-1/2+\epsilon}). \end{aligned} \quad (2.53)$$

Here we have used that since \tilde{P} does not contain replicas 1 and 2 we can replace $n+1, n+2 \rightarrow 1, 2$. Now we apply formula (2.30) using the following relations:

$$\begin{aligned} E\{\langle T_{l, l'} \tilde{P} \rangle\} &= A \sum_{k=2} \delta_{l, 2k-1} \delta_{l', 2k} E\{\dot{q}^{k-2}\} \quad (l < l'), \\ E\{\langle T_l \tilde{P} \rangle\} &= BE\{\dot{q}^{k-2}\}, \\ \sqrt{N}E\{\langle (R_{1, 1} - R_{l, l}) \tilde{P} \rangle\} &= -\tilde{B}E\{\dot{q}^{k-2}\}. \end{aligned}$$

These relations follows from Lemmas 3, 4 and formula (2.50).

$$\begin{aligned} \alpha^{-1}I &= a_1^{(3)} E\{\langle (R_{1, 1} - R) \dot{q}^{k-2} \rangle\} \sqrt{N} \\ &\quad + a_2^{(3)} E\{\dot{q}^k\} + (k-1)a_3^{(3)} E\{\dot{q}^{k-2}\} + O(N^{-1/2+\epsilon}). \end{aligned} \quad (2.54)$$

So we have got the equation

$$\begin{aligned} E\{\dot{q}^k\} &= (k-1)(q^2 + \alpha a_3^{(3)}) E\{\dot{q}^{k-2}\} \\ &\quad + \alpha a_1^{(3)} E\{\langle (R_{1, 1} - R) \dot{q}^{k-2} \rangle\} \sqrt{N} + \alpha a_2^{(3)} E\{\dot{q}^k\} + O(N^{-1/2+\epsilon}), \end{aligned} \quad (2.55)$$

where $a_{1,2,3}^{(3)}$ are some algebraic combinations of $b_{0,1}$ and $c_{0,1,2,3,4}$ defined by (2.35).

By the same way, studying $\sqrt{N}E\{\langle (R_{1, 1} - R) \dot{q}^{k-1} \rangle\}$, we get the equation

$$\begin{aligned} \sqrt{N}E\{\langle (R_{1, 1} - R) \dot{q}^{k-1} \rangle\} &= (k-1)(q^2 + \alpha a_3^{(4)}) E\{\dot{q}^{k-2}\} \\ &\quad + \alpha a_1^{(4)} E\{\langle (R_{1, 1} - R) \dot{q}^{k-2} \rangle\} \sqrt{N} \\ &\quad + \alpha a_2^{(4)} E\{\dot{q}^k\} + O(N^{-1/2+\epsilon}), \end{aligned} \quad (2.56)$$

where $a_{1,2,3}^{(4)}$ are some algebraic combinations of $b_{0,1}$ and $c_{0,1,2,3,4}$ defined by (2.35).

Considering (2.55) and (2.56) as a system of equations with respect to $E\{q^k\}$ and $\sqrt{N}E\{\langle(R_{1,1} - R)\rangle q^{k-1}\}$, we finish the proof of Lemma 5.

3 Auxiliary results

Proof of Proposition 2 Consider any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^N$ and write

$$(\mathcal{A}^{(f)} \mathbf{x}, \mathbf{y}) = \langle (\dot{\mathbf{J}}, \mathbf{x})(\dot{\mathbf{J}}, \mathbf{y})f \rangle \leq \langle (\dot{\mathbf{J}}, \mathbf{x})^2 (\dot{\mathbf{J}}, \mathbf{y})^2 \rangle^{1/2} \langle f^2 \rangle^{1/2} \leq \frac{3\langle f^2 \rangle^{1/2}}{z} |\mathbf{x}| |\mathbf{y}|,$$

where we have used inequality (1.16). Similarly, using inequalities (1.16) and (1.17), for any $\mathbf{x} \in \mathbf{R}^p, \mathbf{y} \in \mathbf{R}^N$

$$\begin{aligned} (\mathcal{B}^{(f)} \mathbf{x}, \mathbf{y}) &= \langle (\dot{\mathbf{J}}, \mathbf{x})^4 \rangle^{1/4} \langle f^4 \rangle^{1/4} \left\langle \sum_{\mu} (g'(S_{\mu}) - \langle g'(S_{\mu}) \rangle)^2 y_{\mu}^2 \right\rangle^{1/2} \\ &\leq \frac{\langle 3f^4 \rangle^{1/4} \|\mathcal{A}_*\|^{1/2} \langle |g''|^2 \rangle^{1/2}}{z} |\mathbf{x}| |\mathbf{y}|. \end{aligned}$$

The inequality for the matrix \mathcal{C} can be proven by the same way.

Proof of Proposition 3

From (1.18) one can easily derive that

$$(g'(S))^2 \leq C_1 - C_2 g(S).$$

Thus it is enough to prove that

$$\left\langle -\frac{1}{N} \sum g(S_{\mu}) \right\rangle \leq C.$$

Define the Hamiltonian

$$-\tilde{\mathcal{H}}(\tau) = \tau \sum g(S_{\mu}) + z(\mathbf{J}, \mathbf{J}).$$

Let $\langle \dots \rangle_{\tau}$ be a corresponding Gibbs average. One can see easily that

$$\varphi(\tau) \equiv -\left\langle \frac{1}{N} \sum g(S_{\mu}) \right\rangle.$$

is a decreasing function of τ . Thus

$$\begin{aligned} \left\langle -\frac{1}{N} \sum g(S_{\mu}) \right\rangle &= \varphi(1) \leq \varphi(0) = \left\langle -\frac{1}{N} \sum g(S_{\mu}) \right\rangle_0 \\ &\leq \frac{C}{N} \langle \sum \langle S_{\mu}^2 \rangle_0 \rangle \leq \frac{C}{zN^2} \sum \langle \xi^{(\mu)}, \xi^{(\mu)} \rangle. \end{aligned}$$

Proof of Lemma 2

We prove first that

$$\begin{aligned} |\nu_t(\tilde{R}_{1,2}) - \nu_0(\tilde{R}_{1,2})| &\leq \frac{C}{\sqrt{N}}, \quad |\nu_t(\tilde{U}_1) - \nu_0(\tilde{U}_1)| \leq \frac{C}{\sqrt{N}}, \\ |\varphi_t(R_{1,2}) - \varphi_0(R_{1,2})| &\leq \frac{C}{\sqrt{N}}, \quad |\varphi_t(R_{1,1}) - \varphi_0(R_{1,1})| \leq \frac{C}{\sqrt{N}}. \end{aligned} \tag{3.1}$$

To this end consider the Hamiltonian $H_N(t)$ which has the form (2.16) with d, U substituted by

$$d_N = \nu_0(g'(S_{\mu}^{(1)})g'(S_{\mu}^{(2)})), \quad U_N = \nu_0(g''(S_{\mu}) + g'^2(S_{\mu}))$$

respectively. Then we use formula (2.19) for $f = \sqrt{N}\tilde{R}_{1,2}$ and $f = \sqrt{N}\tilde{U}_1$, but we write it in the form:

$$\begin{aligned} \nu'_t(f) &= \frac{1}{2} \sum_{l=1}^n \nu_t((f - \langle f \rangle_t) s_l^2 (U_l^- - U_N)) + \sum_{l < l'}^n \nu_t((f - \langle f \rangle_t) s_l s_{l'} (\tilde{R}_{l,l'}^- - d_N)) \\ &\quad - n \sum_{l=1}^n \nu_t((f - \langle f \rangle_t) s_l s_{n+1} (\tilde{R}_{l,n+1}^- - d_N)) \\ &\quad + \frac{n(n+1)}{2} \nu_t((f - \langle f \rangle_t) s_{n+1} s_{n+2} (\tilde{R}_{n+1,n+2}^- - d_N)). \end{aligned} \quad (3.2)$$

Using the Schwartz inequality and (1.17), due to the terms $\nu_t((f - \langle f \rangle_t)^2)$ we obtain the first line of (3.1). But then, on the basis of (3.1) one can derive from (3.2) that the first line of (3.1) is valid even if we replace $CN^{-1/2}$ by CN^{-1} . Now similarly to (2.15) one can conclude that for any $r > 2$

$$E\{|\tilde{R}_{l,l'} - d_N|^r\} \leq \frac{C}{N}, \quad E\{|\tilde{U}_l - U_N|^r\} \leq \frac{C}{N}. \quad (3.3)$$

Similarly, using (2.27) for $f = \sqrt{N}R_{1,2}$ and $f = \sqrt{N}R_{1,1}$ with $q_N = \phi_0(s_1 s_2)$, $R_N = \phi_0(s_1^2)$, we prove first the second line of (3.1). Then, by the same way as above, we get

$$E\{|\langle R_{l,l} - q_N \rangle|^2\} \leq \frac{C}{N}, \quad E\{|\langle R_{l,l} - R_N \rangle|^2\} \leq \frac{C}{N}. \quad (3.4)$$

Now we remark that since it was proved in [S-T2], that the system (1.11) has a unique solution, to prove (2.11) it is enough to show that our q_N , R_N satisfy this system with the error terms $O(N^{-1+\epsilon})$ and d_N , U_N satisfy relations (2.10) with the same error. Now, on the basis of (3.3) and (3.4) it can be shown easily by formulas (2.19), (2.21) with $f(s) = s_1 s_2$ and $f(s) = s_1^2$ and by formulas (2.27) and (2.30) with $f = \mathcal{D}_1 \mathcal{D}_2$ and $f = \mathcal{D}_1^2$.

Proof of Proposition 4

Let us denote

$$\begin{aligned} \phi(s) &= \log \int d\mathbf{J}^- \exp\{-H_t(\mathbf{J}^-, s)\} = \phi_t(s) + us\sqrt{d(1-t)} - \frac{s^2}{2}(z - (1-t)(U-d)) \\ \Rightarrow \langle s^n \rangle_t &= \langle s^n \rangle_\phi = \frac{\int s^n e^{\phi(s)} ds}{\int e^{\phi(s)} ds}. \end{aligned} \quad (3.5)$$

According to the results [1], $\phi_t(s)$ is a concave function. Besides, there exists $\delta > 0$ such that

$$(z - (1-t)(U-d)) \geq (z - (U-d)) = (R-q)^{-1} > \delta.$$

Then, according to the results [1],

$$\langle |s - \langle s \rangle_\phi|^n \rangle_\phi \leq \delta^{-n} 2^n \Gamma(n). \quad (3.6)$$

So, to prove Proposition 4 it is enough to estimate $\langle s \rangle_\phi$.

Denote s^* the point of maximum of the function $\phi(s)$. Then it follows from representation (3.5) of the function $\phi(s)$ that

$$|s^*| \leq \delta^{-1} |\varphi'_t(0) + u\sqrt{d(1-t)}|.$$

On the other hand, by [1]

$$\langle |s - s^*| \rangle_\phi \leq \delta^{-1} \Rightarrow |\langle s \rangle_\phi| \leq \delta^{-1} (1 + |\phi'_t(0) + \sqrt{d(1-t)}u|)$$

Now, since

$$\phi'_t(0) = \sum_{\mu} \frac{\xi_1^{(\mu)}}{\sqrt{N}} \langle g'(S_{\mu}^-) \rangle_0$$

and $\langle \dots \rangle_0$ does not depend on $\xi_1^{(\mu)}$, we have

$$E\{\langle s \rangle^n\} \leq 2^n \delta^{-n} \Gamma(n) \left(d^n + E \left\{ \left(N^{-1} \sum_{\mu} \langle g'(S_{\mu}^-) \rangle_0^2 \right)^{n/2} \right\} \right).$$

Then using Proposition 3, we obtain the statement of Proposition 4.

Proof of Proposition 5

According to the representation (2.26), \mathcal{P}_t is some polynomial of the derivatives $\frac{\partial^k}{\partial S^k} \log G_t(S_1^{(j)}, u)$ ($k = 1, \dots, 6$, $j = 1, \dots, n$). But under condition (1.18) for $k \geq 2$ these derivatives are uniformly bounded functions. So we need only to prove that

$$E \left\{ \left\langle \left(\frac{\partial}{\partial S} \log G_t(S_1^{(j)}, u) \right)^{2k} \right\rangle_{(t)} \right\} \leq C(k). \quad (3.7)$$

Similarly to the proof of Proposition 4 by (1.17) and (1.18) inequality (3.7) can be derived from the inequality

$$E \left\{ \left\langle \left(\frac{\partial}{\partial S} \log G_t(S_1^{(j)}, u) \right)^2 \right\rangle_{(t)}^k \right\} \leq C(k). \quad (3.8)$$

But similarly to the proof of Proposition 3, since

$$\left(\frac{\partial}{\partial S} \log G_t(S_1^{(j)}, u) \right)^2 \leq C_1 - C_2 \log G_t(S_1^{(j)}, u),$$

one can get that

$$\left\langle \left(\frac{\partial}{\partial S} \log G_t(S_1^{(j)}, u) \right)^2 \right\rangle_{(t)} \leq C_1 - C_2 \left\langle \log G_t(S_1^{(j)}, u) \right\rangle_{(0)}.$$

Now, since $\langle \dots \rangle_{(0)}$ does not depend on $\{\xi_1^{(\mu)}\}_{\mu=1}^p$, (3.8) follows immediately.

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